A UNIFORM NILSEQUENCE WIENER–WINTNER THEOREM FOR BILINEAR ERGODIC AVERAGES

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Abstract. We show that a k-linear pointwise ergodic theorem on an ergodic measure-preserving system implies a uniform k-linear nilsequence Wiener–Wintner theorem on that system. The assumption is known to hold for arbitrary systems and $k=2$ (due to Bourgain) and for distal systems and arbitrary k (due to Huang, Shao, and Ye).

1. Introduction

Let (X,μ,T) be an ergodic measure-preserving system and Φ a Følner sequence in $\mathbb{Z}$. Call a sequence $(a_n)_n$ a *good weight for the k-linear pointwise ergodic theorem on X along* Φ if for some distinct non-zero integers $b_1,\dots,b_k$ and any bounded functions $f_1,\dots,f_k \in L^\infty(X)$ the limit

$$\lim_{N\to\infty} \frac{1}{|\Phi_N|} \sum_{n\in\Phi_N} a_n \prod_{i=1}^{k} f_i(T^{b_i n}x)$$

exists pointwise almost everywhere. With this terminology, the nilsequence Wiener–Wintner theorem [HK09, Theorem 2.22] tells that nilsequences are good weights for the 1-linear pointwise ergodic theorem on any measure-preserving system along the standard Følner sequence $\Phi_N = \{1,\dots,N\}$. The 1-linear pointwise ergodic theorem, that is, the fact that $a_n \equiv 1$ is a good weight, has been used as a black box in its proof. Recently, Assani, Duncan, and Moore [ADM14] showed that the sequences $a_n = e^{ip(n)}$, p polynomial, are good weights for the 2-linear pointwise ergodic theorem, similarly using the $a_n \equiv 1$ case as a black box. Moreover, their result is uniform in the same way as the 1-linear nilsequence Wiener–Wintner theorem in [EZK13].

In this note we prove the natural joint generalization of these results. We fix a Følner sequence Φ and say that the system (X,μ,T) has property P_k if $a_n \equiv 1$ is a good weight for the k-linear pointwise ergodic theorem on X along Φ. It is a long-standing conjecture that every measure-preserving system satisfies P_k for every k along the standard Følner sequence $\Phi_N = \{1,\dots,N\}$, and we have nothing to add on this issue. For $k=2$ this conjecture has been proved by Bourgain [Bou90] (see also [Dem07] and [DOP15]). Some partial results for $k>2$ can be found in [HSY14] and [Ass98].

Our main result is the following uniformity seminorm estimate. We refer to the prequel [EZK13] for definitions of various concepts related to nilmanifolds G/Γ and to [ZK15] for the definition of the modified uniformity seminorms $U^{k+l}(T,c)$.

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Theorem 1.1. *Suppose that the ergodic measure-preserving system (X,μ,T) has property P_k with parameters $b_1,\dots,b_k$ along the Følner sequence Φ. Then*

(1.2)
$$\int \limsup_{N\to\infty} \sup_{G/\Gamma : G_\bullet \text{ has length } l} C_{G/\Gamma}^{-1} \sup_{\substack{g\in P(\mathbb{Z},G_\bullet)\\ F\in W^{r,2^l}(G/\Gamma)}} \Big| \|F\|_{W^{r,2^l}(G/\Gamma)}^{-1} \frac{1}{|\Phi_N|} \sum_{n\in\Phi_N} \prod_{i=1}^{k} f_i(T^{b_i n}x) F(g(n)\Gamma) \Big|^{2^{l+1}} \lesssim_{b_1,\dots,b_k,l} \min_i \|f_i\|_{U^{k+l}(T,c)}^{2^{l+1}},$$

where $r=\sum_{m=1}^{l}(d_m-d_{m+1})\binom{l}{m-1}$ with $d_i=\dim G_i$, $c=c(b_1,\dots,b_k)$, and the positive constants $C_{G/\Gamma}$ depend only on the nilmanifold G/Γ, that is, the filtration $G_\bullet$, the lattice Γ, and the Mal'cev basis used to define the Sobolev spaces $W^{r,2^l}(G/\Gamma)$.

The case $k=2$, $l=1$ has been proved by Assani, Duncan, and Moore [ADM14], and the case of commutative G by Assani and Moore [AM14a; AM14b]. An immediate consequence of Theorem 1.1 is a nilsequence Wiener–Wintner theorem. Before formulating it let us recall the following fact.

Proposition 1.3. *Let $(c_n)_{n\in\mathbb{Z}}$ be a bounded complex-valued sequence. Then the following statements are equivalent.*

(1) *For every nilsequence (a_n) the averages*
$$\frac{1}{N}\sum_{n=1}^{N} a_n c_n$$
converge as $N\to\infty$.

(2) *The sequence (c_n) is a good weight for polynomial multiple ergodic averages along $\{1,\dots,N\}$, i.e., for every measure-preserving system (Y,ν,S), integer polynomials $p_1,\dots,p_k$, and functions $f_1,\dots,f_k\in L^\infty(Y,\nu)$ the averages*
$$\frac{1}{N}\sum_{n=1}^{N} \phi(T^n x) S^{p_1(n)} f_1 \cdots S^{p_k(n)} f_k$$
converge in $L^2(Y,\nu)$ as $N\to\infty$.

(3) *The sequence (c_n) is a good weight for linear multiple ergodic averages along $\{1,\dots,N\}$, i.e.,* (2) *holds with $p_i(n)=b_i n$, b_i arbitrary.*

Proof. The implication (1) $\implies$ (2) is [Chu09, Theorem 1.3] and the implication (2) $\implies$ (3) is immediate. Finally, the implication (3) $\implies$ (1) follows from [Fra14, Proposition 2.4]. □

We can now formulate our Wiener–Wintner theorem.

Corollary 1.4. *Suppose that the ergodic measure-preserving system (X,μ,T) has property P_k with parameters $b_1,\dots,b_k$ along the standard Følner sequence $\Phi_N=\{1,\dots,N\}$. Then for every functions $f_1,\dots,f_k\in L^\infty(X,\mu)$ there exists a full measure set $X'\subset X$ such that for every $x\in X'$ the sequence*
$$\Big(\prod_{i=1}^{k} f_i(T^{b_i n}x)\Big)_n$$
satisfies the equivalent properties stated in Proposition 1.3.

This follows from Theorem 1.1 using the characterization (1) in Proposition 1.3 in a standard way (see e.g. [EZK13, §6] and use [ZK15, Lemma 2.6]). By Bourgain's bilinear pointwise ergodic theorem Corollary 1.4 holds unconditionally for $k=2$. In this case Corollary 1.4 has been previously proved in [AM15, Theorem 1.4] and reproved in [Ass15] after the appearance of this note.

2. Proof of Theorem 1.1

By induction on l. For $l=0$ the group G is trivial, so the nilsequences are constant and the averages over n converge pointwise almost everywhere by property P_k. Hence the left-hand side of (1.2) equals

$$\Big\| \lim_{N\to\infty} \frac{1}{|\Phi_N|} \sum_{n\in\Phi_N} \prod_{i=1}^{k} f_i(T^{b_i n}x) \Big\|_{L^2_x}^2,$$

the limit now being taken in L^2. The uniformity seminorm estimate for this limit originates in [HK05, Theorem 12.1]; the version used here can be found in [ZK15, Lemma 2.4].

Suppose now that the claim holds for $l-1$. Writing the function F as a vertical Fourier series $F=\sum_\chi F_\chi$ as in [EZK13, (3.5)] we obtain for the supremum on the left-hand side of (1.2)

$$\sup_{\substack{g\in P(\mathbb{Z},G_\bullet)\\ F\in W^{r,2^l}(G/\Gamma)}} \Big| \|F\|^{-1}_{W^{r,2^l}(G/\Gamma)} \frac{1}{|\Phi_N|} \sum_{n\in\Phi_N} \prod_{i=1}^{k} f_i(T^{b_i n}x) F(g(n)\Gamma) \Big|^{2^{l+1}}$$

$$\leq \sup_{\substack{g\in P(\mathbb{Z},G_\bullet)\\ F\in W^{r,2^l}(G/\Gamma)}} \Big| \sum_\chi \frac{\|F_\chi\|_{W^{r-d_l,2^l}(G/\Gamma)}}{\|F\|_{W^{r,2^l}(G/\Gamma)}} \|F_\chi\|^{-1}_{W^{r-d_l,2^l}(G/\Gamma)} \frac{1}{|\Phi_N|} \sum_{n\in\Phi_N} \prod_{i=1}^{k} f_i(T^{b_i n}x) F_\chi(g(n)\Gamma) \Big|^{2^{l+1}}$$

$$\lesssim_{G/\Gamma} \sup_{\substack{g\in P(\mathbb{Z},G_\bullet)\\ F\in W^{r-d_l,2^l}(G/\Gamma)\text{ vertical character}}} \Big| \|F\|^{-1}_{W^{r-d_l,2^l}(G/\Gamma)} \frac{1}{|\Phi_N|} \sum_{n\in\Phi_N} \prod_{i=1}^{k} f_i(T^{b_i n}x) F(g(n)\Gamma) \Big|^{2^{l+1}},$$

where we have used [EZK13, Lemma 3.7] in the last line. We incorporate the G/Γ-dependent constant into $C_{G/\Gamma}$, which may change from line to line. Next we apply the van der Corput lemma [EZK13, Lemma 2.7] and estimate the integrand on the left-hand side of (1.2) by

$$\text{C-}\liminf_m \limsup_{N\to\infty} \sup_{G/\Gamma} C^{-1}_{G/\Gamma} \sup_{\substack{g\in P(\mathbb{Z},G_\bullet)\\ F\in W^{r-d_l,2^l}(G/\Gamma)\text{ vertical character}}} \Big| \|F\|^{-2}_{W^{r-d_l,2^l}(G/\Gamma)} \frac{1}{|\Phi_N|} \sum_{n\in\Phi_N} \prod_{i=1}^{k} f_i(T^{b_i n}x) F(g(n)\Gamma) \cdot \prod_{i=1}^{k} \overline{f_i(T^{b_i(n+m)}x) F(g(n+m)\Gamma)} \Big|^{2^l},$$

where $\text{C-}\liminf_m a_m := \liminf_{M\to\infty} \Big|\frac{2}{M^2}\sum_{m=-M}^{M}(M-|m|)a_m\Big|$. With the notation for the cube construction from [EZK13, § 3] this becomes

$$\text{C-}\liminf_m \limsup_{N\to\infty} \sup_{G/\Gamma} C^{-1}_{G/\Gamma} \sup_{\substack{g\in P(\mathbb{Z},G_\bullet)\\ F\in W^{r-d_l,2^l}(G/\Gamma)\text{ vertical character}}} \Big| \|F\|^{-2}_{W^{r-d_l,2^l}(G/\Gamma)} \frac{1}{|\Phi_N|} \sum_{n\in\Phi_N} \prod_{i=1}^{k} (\overline{f_i} T^{b_i m} f_i)(T^{b_i n}x) \tilde{F}_m(\tilde{g}_m(n)\tilde{\Gamma}) \Big|^{2^l}.$$

By [EZK13, Lemma 3.2] this is bounded by

$$\text{C-}\liminf_m \limsup_{N\to\infty} \sup_{G/\Gamma} C^{-1}_{\tilde{G}/\tilde{\Gamma}} \sup_{\substack{\tilde{g}_m\in P(\mathbb{Z},\tilde{G}_\bullet)\\ \tilde{F}_m\in W^{r-d_l,2^{l-1}}(\tilde{G}/\tilde{\Gamma})}} \Big| \|\tilde{F}_m\|^{-1}_{W^{r-d_l,2^{l-1}}(\tilde{G}/\tilde{\Gamma})} \frac{1}{|\Phi_N|} \sum_{n\in\Phi_N} \prod_{i=1}^{k} (\overline{f_i} T^{b_i m} f_i)(T^{b_i n}x) \tilde{F}_m(\tilde{g}_m(n)\tilde{\Gamma}) \Big|^{2^l}.$$

Integrating the last display over X and applying Fatou's lemma and the inductive hypothesis we obtain

$$\text{C-}\liminf_m \min_i \|\overline{f_i} T^{b_i m} f_i\|^{2^l}_{U^{k+l-1}(T,c)}.$$

By Hölder's inequality and the inductive construction of the Gowers–Host–Kra seminorms this is bounded by

$$\begin{aligned}&\min_i \text{C-}\liminf_m \|\overline{f_i} T^{b_i m} f_i\|_{U^{k+l-1}(T,c)}^{2^l}\\ &\leq \min_i \big(\text{C-}\liminf_m \|\overline{f_i} T^{b_i m} f_i\|_{U^{k+l-1}(T,c)}^{2^{k+l-1}} \big)^{2^{-k+1}}\\ &\lesssim \min_i \|f_i\|_{U^{k+l}(T,c)}^{2^{l+1}}\end{aligned}$$

as required.

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